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Feynman Path-Integral Calculation of the Polaron Effective Mass*

John T. Marshall and M. S. Chawla
Department of Physics and Astronomy, Louisiana State University, Baton Rouge, Louisiana 70803
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In the expansion of the ground-state energy of a polaron in a weak magnetic field, the zeroth-order term is the polaron self-energy, while the first-order term is inversely proportional to the polaron effective mass. The effective mass so obtained is exactly equivalent to the free-polaron effective mass as defined by Fröhlich. This equivalence principle is used to approximate the polaron effective mass by employing an approximate expression for the ground-state energy of a polaron in a weak magnetic field obtained by applying Feynman's path-integral variational method. The resultant polaron effective mass is found to be higher than Feynman's result by less than 1%.

In a description of the motion of a single conduction electron in an ionic crystal, Fröhlich[1] developed a model which depicts the electron as a Bloch electron of crystal mass m, interacting with the polarization field which arises from the long-wavelength longitudinal optical crystal modes characterized by a single frequency ω. The strength of this interaction is determined by a dimensionless coupling constant

$$\alpha = \tfrac{1}{2}(\epsilon_\infty^{-1} - \epsilon^{-1})(e^2/\hbar\omega)(2m\omega/\hbar)^{1/2}, \tag{1}$$

where ϵ and ϵ_∞ are the static and optical dielectric constants of the crystal. The composite entity consisting of the electron together with its accompanying nonradiative polarization field is called the polaron. The total wave vector, $\vec{K}$, of the system consisting of the electron and the polarization field is a constant of the motion. Based on this exact conservation law, Fröhlich defined the self-energy $E_0(\alpha)$ and the effective mass $\mu(\alpha)$ of the polaron by the expansion formula[2]

$$E_0(\alpha, \vec{k}) = E_0(\alpha) + \tfrac{1}{2}\mu(\alpha)^{-1}k^2 + O(k^4), \tag{2}$$

where $E_0(\alpha, \vec{k})$ is the system's least energy eigenvalue whose corresponding eigenfunction is simultaneously an eigenfunction of $\vec{K}$ with corresponding eigenvalue $\vec{k}$.

Various methods have been employed to approximate the self-energy and the effective mass of the polaron.[3,4] The weak-coupling result of Lee, Low, and Pines[5] yields

$$E_0(\alpha) = -\alpha, \tag{3}$$

$$\mu(\alpha) = 1 + \tfrac{1}{6}\alpha. \tag{4}$$

These results are exactly correct to first order in α, in the limit of weak coupling. The strong-coupling variational approximation of Landau[6] and Pekar[7] (LP) yields

$$E_0(\alpha) = -a\alpha^2, \tag{5}$$

$$\mu_{\rm LP}(\alpha) = b\alpha^4 + 1, \tag{6}$$

with

$$a \approx 0.10, \tag{7}$$

$$b \approx 0.02. \tag{8}$$

The analytic form of Eq. (5) is asymptotically correct.[8] The effective mass given by Eq. (6) is based upon an alternative definition of the effective mass which is equivalent to Fröhlich's definition only when the trial wave function employed is an exact eigenfunction of the total wave vector $\vec{K}$. The trial wave function employed in the Landau-Pekar approximation, however, is not such an eigenfunction. The Feynman-Schultz[9,10] results for the polaron self-energy and effective mass agree with the Lee-Low-Pines results for weak coupling, agree with the Landau-Pekar results for strong coupling, and possess smooth transitional behavior for intermediate coupling. Feynman's method, however, does not incorporate conservation of the total wave vector and, consequently, the Feynman-Schultz effective-mass calculation is based upon still another definition.

It is the purpose of this paper to calculate Fröhlich's polaron effective mass by using the Feynman path-integral approach. To accomplish this purpose, the ground-state energy of a polaron in a weak magnetic field is calculated using the same method as Feynman uses to calculate the ground-state energy of a free polaron. Use is then made of a theorem proved by Marshall and Roberts,[11] which states that the exact ground-state energy of a polaron in a weak magnetic field $\vec{B}$ may be expanded in the form

$$E_0(\alpha, \lambda) = E_0(\alpha) + \tfrac{1}{2}\mu(\alpha)^{-1}\lambda + O(\lambda^2), \tag{9}$$

where

$$\lambda = eB/c, \tag{10}$$

and where $E_0(\alpha)$ and $\mu(\alpha)$ are exactly the self-energy and effective mass of a free polaron as defined by Fröhlich.

By following Feynman's method,[9] one may write the exact ground-state energy of the polaron in a magnetic field in terms of a Feynman path integral as

$$E_0(\alpha,\lambda) = -\lim_{T\to\infty} T^{-1}\ln\left[\int_{\vec{0},0}^{\vec{0},T} e^{S} D\vec{r}(t)\right], \tag{11}$$

where

$$S = \int_0^T \left(-\tfrac{1}{2}\dot{\vec{r}}^{\,2} + i\lambda y\dot{x}\right)dt + 8^{-1/2}\alpha\int_0^T\int_0^T e^{-|t-s|}\left[\,|\vec{r}(t)-\vec{r}(s)|\,\right]^{-1}dt\,ds, \tag{12}$$

where the path integral is over all paths $\vec{r}(t)$ satisfying the boundary conditions $\vec{r}(0)=\vec{r}(T)=\vec{0}$, and where x and y denote the components of $\vec{r}$ perpendicular to the direction of $\vec{B}$. This expression for the ground-state energy reduces to Feynman's expression for a free polaron for $\lambda = 0$. By the Feynman method, the term in S containing λ is obtained by replacing the time variable t by $-it$ in the magnetic-field-dependent term of the action integral for a polaron in a magnetic field.

It has not been found possible to evaluate the path integral in Eq. (11). For a weak magnetic field, one may obtain an upper-bound approximation for $E_0(\alpha,\lambda)$ by use of an extension of Feynman's variational principle discussed in the Appendix. According to this principle, for a sufficiently weak magnetic field,

$$E_0(\alpha,\lambda) \le E_0'(\alpha,\lambda), \tag{13}$$

where

$$E_0'(\alpha,\lambda) = E_0^{(0)}(\alpha,\lambda) - \lim_{T\to\infty} T^{-1}\langle S - S'\rangle, \tag{14}$$

where S' purports to approximate S and is given by

$$S' = \int_0^T \left(-\tfrac{1}{2}\dot{\vec{r}}^{\,2} + i\lambda y\dot{x}\right)dt - \tfrac{1}{2}C\int_0^T\int_0^T e^{-w|t-s|}\left[\vec{r}(t)-\vec{r}(s)\right]^2 dt\,ds, \tag{15}$$

where C and w are variational parameters chosen to minimize $E_0'(\alpha,\lambda)$. In Eq. (14),

$$E_0^{(0)}(\alpha,\lambda) = -\lim_{T\to\infty} T^{-1}\ln\int_{\vec{0},0}^{\vec{0},T} e^{S'} D\vec{r}(t); \tag{16}$$

the angle brackets denote the path average defined by

$$\langle f[\vec{r}(t)]\rangle = \int_{\vec{0},0}^{\vec{0},T} e^{S'} f[\vec{r}(t)]D\vec{r}(t) \Big/ \int_{\vec{0},0}^{\vec{0},T} e^{S'} D\vec{r}(t), \tag{17}$$

where $f[\vec{r}(t)]$ represents a function of path. The expression for $E_0'(\alpha,\lambda)$ given by Eq. (14) approximates $E_0(\alpha,\lambda)$ in the respects that Eq. (14) can be regarded as an expansion for $E_0(\alpha,\lambda)$ correct through first order in $(S-S')$ and that the variational principle may be brought to bear in order to optimize the result.

For the purpose of evaluating $E_0'(\alpha,\lambda)$ it is convenient to employ the identity

$$|\vec{r}(t)-\vec{r}(s)|^{-1} = \int (2\pi^2k^2)^{-1} \times\exp\left[\int_0^T \vec{f}(\vec{k},t,\tau,\sigma)\cdot\vec{r}(t)\,dt\right]d^3\vec{k}, \tag{18}$$

where

$$\vec{f}(\vec{k},t,\tau,\sigma) = i\vec{k}\left[\delta(t-\tau)-\delta(t-\sigma)\right]; \tag{19}$$

and the definition

$$W(\vec{k},\tau,\sigma) \equiv \left\langle \exp\left[\int_0^T \vec{f}(\vec{k},t,\tau,\sigma)\cdot\vec{r}(t)\,dt\right]\right\rangle \tag{20}$$

to obtain

$$E_0'(\alpha,\lambda) = E_0^{(0)}(\alpha,\lambda) - (A+B), \tag{21}$$

where

$$A = \lim_{T\to\infty} \alpha T^{-1}8^{-1/2}\int_0^T\int_0^T e^{-|\tau-\sigma|} \times\langle\, |\vec{r}(\tau)-\vec{r}(\sigma)|^{-1}\rangle\, d\tau\, d\sigma = \lim_{T\to\infty} \alpha T^{-1}8^{-1/2}\int_0^T\int_0^T e^{-|\tau-\sigma|} \times\int (2\pi^2k^2)^{-1}W(\vec{k},\tau,\sigma)\,d^3\vec{k}\,d\tau\,d\sigma, \tag{22}$$

$$B = \lim_{T\to\infty} \tfrac{1}{2}CT^{-1}\int_0^T\int_0^T e^{-w|\tau-\sigma|} \times\langle\, |\vec{r}(\tau)-\vec{r}(\sigma)|^2\rangle\, d\tau\, d\sigma = \lim_{T\to\infty} \tfrac{1}{2}CT^{-1}\int_0^T\int_0^T e^{-w|\tau-\sigma|} \times\left[-\nabla_{\vec{k}}^2 W(\vec{k},\tau,\sigma)\big|_{\vec{k}=0}\right]d\tau\, d\sigma, \tag{23}$$

$$E_0^{(0)}(\alpha,\lambda) = \tfrac{1}{2}\lambda + \int_0^C C^{-1}B(C,w)\,dC. \tag{24}$$

The last equation is obtained by differentiation of Eqs. (15) and (16) and by use of Eq. (23) and the boundary condition that for $C=0$, $E_0^{(0)}(\alpha,\lambda)$ reduces to the ground-state energy $\frac{1}{2}\lambda$ of the polaron in the absence of interaction with the polarization field. It is clear from Eqs. (21)–(24) that in order to obtain a tenable formula for $E_0'(\alpha,\lambda)$, a simplified expression is needed for $W(\vec{k},\tau,\sigma)$, which may be written out by the use of Eqs. (17) and (20) as

$$W(\vec{k},\tau,\sigma) = \int_{\vec{0},0}^{\vec{0},T} \exp\left\{\left[\int_0^T \vec{f}(\vec{k},t,\tau,\sigma)\cdot\vec{r}(t)\,dt\right] + S'\right\} D\vec{r}(t) \Big/ \int_{\vec{0},0}^{\vec{0},T}\exp(S')\,D\vec{r}(t). \tag{25}$$

This expression may be simplified by changing the path-integration variable $\vec{r}(t)$ in the numerator of Eq. (25) to a new variable $\vec{r}'(t) \equiv \vec{r}(t) - \bar{\vec{r}}(t)$, where $\bar{\vec{r}}(t)$ is that path for which the exponent in the numerator of Eq. (25) is extremal and for which $\vec{r}(0) = \vec{r}(T) = \vec{0}$. The resultant numerator contains the denominator as a factor, with the consequence that

$$W(k,\tau,\sigma) = \exp\left\{\tfrac{1}{2}\int_0^T \vec{f}(\vec{k},t,\tau,\sigma)\cdot\bar{\vec{r}}(t)\,dt\right\}. \tag{26}$$

Extremization of the exponent of the numerator of Eq. (25) yields the following integrodifferential equations for the components of $\bar{\vec{r}}(t)$:

$$\ddot{\bar{x}}(t) = 2C\int_0^T e^{-w|t-s|}[\bar{x}(t)-\bar{x}(s)]ds + i\lambda\dot{\bar{y}} - f_x(\vec{k}, t, \tau, \sigma), \quad (27)$$

$$\ddot{\bar{y}}(t) = 2C\int_0^T e^{-w|t-s|}[\bar{y}(t)-\bar{y}(s)]ds - i\lambda\dot{\bar{x}} - f_y(\vec{k}, t, \tau, \sigma), \quad (28)$$

$$\ddot{\bar{z}}(t) = 2C\int_0^T e^{-w|t-s|}[\bar{z}(t)-\bar{z}(s)]ds - f_z(\vec{k}, t, \tau, \sigma), \quad (29)$$

where f_x, f_y, and f_z are the components of $\vec{f}$ given by Eq. (19). These equations may be solved conveniently in the limit $T\rightarrow\infty$ by the Fourier transform method. Apart from transient terms [which are appreciable only near the end points $t=0$ and $t=T$, and which are irrelevant to the desired evaluation of Eqs. (22) and (23) in the required limit $T\rightarrow\infty$] and apart from constant terms, the results are

$$\bar{x}(t) = \int_0^T G_{xx}(t,t') f_x(\vec{k}, t', \tau, \sigma)dt' + \int_0^T G_{xy}(t,t') \times f_y(\vec{k}, t', \tau, \sigma)dt', \quad (30)$$

$$\bar{y}(t) = \int_0^T G_{yy}(t,t') f_y(\vec{k}, t', \tau, \sigma)dt' + \int_0^T G_{yx}(t,t') \times f_x(\vec{k}, t', \tau, \sigma)dt', \quad (31)$$

$$\bar{z}(t) = \int_0^T G_{zz}(t,t') f_z(\vec{k}, t', \tau, \sigma)dt', \quad (32)$$

where

$$G_{yy}(t,t') = G_{xx}(t,t'), \quad (33)$$

$$G_{xy}(t,t') = -G_{yx}(t,t'), \quad (34)$$

$$G_{zz}(t,t') = -\frac{1}{2\pi}\,\mathrm{PP}\int_{-\infty}^{\infty}\frac{\xi^2+w^2}{\xi^2(\xi^2+v^2)}(1-e^{i\xi|t-t'|})\,d\xi = -\tfrac{1}{2}v^{-2}[v^{-1}(v^2-w^2)(1-e^{-v|t-t'|}) + w^2|t-t'|], \quad (35)$$

$$G_{xx}(t,t') = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{(\xi^2+v^2)(\xi^2+w^2)}{\xi^2(\xi^2+v^2)^2+\lambda^2(\xi^2+w^2)^2}\times(1-e^{i\xi|t-t'|})\,d\xi = G_{zz}(t,t') + \tfrac{1}{4}\lambda(t-t')^2w^4v^{-4} + O(\lambda^2), \quad (36)$$

where PP stands for principal part and

$$v^2 \equiv w^2 + 4C/w. \quad (37)$$

The expansion in Eq. (36) may be obtained by expressing the integral representation of $G_{xx}-G_{zz}$ in terms of the residues of its integrand. The contribution to $G_{xx}-G_{zz}$ from residues which occur at $\xi\neq 0$ in the limit $\lambda\rightarrow 0$ is of order λ^2, and is therefore not required in the present context. The remaining contribution may be evaluated easily to the first order in λ.

Substitution of Eqs. (19) and (30)–(32) into Eqs. (26) and use of Eqs. (33) and (34) yields

$$W(\vec{k}, \tau, \sigma) = \exp[(k_x^2+k_y^2)G_{xx}(|\tau-\sigma|) + k_z^2G_{zz}(|\tau-\sigma|)]. \quad (38)$$

Use of this result for $W(\vec{k}, \tau, \sigma)$ in Eqs. (21)–(24) and performance of the indicated operations yields

$$E_0'(\alpha,\lambda) = E_0^{(0)} - A - B, \quad (39)$$

where

$$E_0^{(0)} = \tfrac{3}{2}(v-w) + \tfrac{1}{2}\lambda w^2v^{-2} + O(\lambda^2), \quad (40)$$

$$A = \alpha\pi^{-1/2}vw^{-1}\int_0^\infty J(u)^{-1/2}e^{-u}du + \tfrac{1}{6}\lambda\alpha\pi^{-1/2}wv^{-1}\int_0^\infty J(u)^{-3/2}u^2e^{-u}du + O(\lambda^2), \quad (41)$$

$$B = \tfrac{3}{4}v^{-1}(v^2-w^2) - \tfrac{1}{2}\lambda w^2v^{-4}(v^2-w^2) + O(\lambda^2), \quad (42)$$

where

$$J(u) = u + v^{-1}w^{-2}(v^2-w^2)(1-e^{-vu}). \quad (43)$$

In obtaining this result, it is helpful to change the integration variables σ and τ occuring in Eqs. (22) and (23) to the new variables $u\equiv|\tau-\sigma|$ and τ.

In view of the inequality (13), the variational parameters v and w are to be determined by minimizing $E_0'(\alpha,\lambda)$. Minimization of $E_0'(\alpha,\lambda)$ as given in expanded form by Eqs. (39)–(43) yields

$$E_0'(\alpha,\lambda) = E_0'(\alpha) + \tfrac{1}{2}\mu'(\alpha)^{-1}\lambda + O(\lambda^2), \quad (44)$$

where

$$E_0'(\alpha) \equiv E_F(\alpha) = \tfrac{3}{4}v^{-1}(v-w)^2 - \alpha\pi^{-1/2}vw^{-1}\int_0^\infty J(u)^{-1/2}e^{-u}du \quad (45)$$

$$\mu'(\alpha) = [1-(1-w^2/v^2)^2 - \tfrac{1}{3}\alpha\pi^{-1/2}wv^{-1}\int_0^\infty J(u)^{-3/2}u^2e^{-u}du]^{-1}, \quad (46)$$

where the values of v and w are just those which minimize Feynman's polaron self-energy $E_F(\alpha)$ and have already been evaluated numerically by Schultz.[10] Comparison of Eq. (44) with Eq. (9) shows that Eqs. (44) and (46) give the polaron self-energy and effective mass in the present approximation. For comparison Feynman's polaron effective mass by be expressed as[9]

$$m_F(\alpha) = 1 + \tfrac{1}{3}\alpha\pi^{-1/2}v^3w^{-3}\int_0^\infty J(u)^{-3/2}u^2e^{-u}du. \quad (47)$$

For small α,[9]

$$v = 3 + \tfrac{2}{9}\alpha + O(\alpha^2), \quad (48)$$

$$w = 3 + O(\alpha), \quad (49)$$

$$m_F(\alpha) = 1 + \tfrac{1}{6}\alpha + \tfrac{72}{2916}\alpha^2 + O(\alpha^3), \quad (50)$$

and consequently, from Eq. (46),

$$\mu'(\alpha) = 1 + \tfrac{1}{6}\alpha + \tfrac{73}{2916}\alpha^2 + O(\alpha^3). \quad (51)$$

For weak coupling, therefore, $m_F(\alpha)$ and $\mu'(\alpha)$ differ only slightly. Similarly, for large α,

$$v = 4\alpha^2/9\pi + O(\alpha^0), \quad (52)$$

$$w = 1 + O(\alpha^{-2}), \quad (53)$$

$$\mu'(\alpha) \sim m_F(\alpha) = 16\alpha^4/81\pi^2 + O(\alpha^2). \quad (54)$$

For intermediate values of α, numerical evaluations are required for the two integrals occuring in Eqs. (45)–(47). A computer program was written to minimize $E_F(\alpha)$ with respect to v and w and to calculate the corresponding result for the polaron self-energy $E_F(\alpha)$ and the effective masses $\mu'(\alpha)$ and $m_F(\alpha)$ as given by Eqs. (46) and (47). The results are shown in Table I. The optimum values of the parameters v and w have already been reported by Schultz,[10] but were recalculated accurately to eight significant digits in order to obtain the percentage difference between $\mu'(\alpha)$ and m_F accurately to two places after the decimal. The results shown in Table I have been rounded off and are accurate to the number of significant figures reported. The values obtained for v and w are in slight disagreement with the values reported by Schultz,[10] but agree with the independent calculation of Marshall and Mills.[12]

In summary, Feynman's method of approximating the ground-state energy of a free polaron has been extended to approximate the ground-state energy of a polaron in a weak magnetic field. The result provides a means of approximating the polaron effective mass based on a definition which is exactly equivalent to Fröhlich's definition of the free-polaron effective mass. The effective mass $\mu'(\alpha)$ thereby obtained is slightly higher (by less than 1%) than Feynman's result, which was based on the same two-parameter model, but which was determined from Feynman's alternative, rather *ad hoc* definition.

The present work is similar to a calculation of the polaron effective mass by Hellwarth and Platzman.[13] Their method, which is based on the same two-parameter model employed here, involves approximating the free energy $F(\theta, \lambda)$ of a polaron in a magnetic field as a function of temperature θ and of the magnetic field strength λ. The free energy has greater informational content than the ground-state energy, but is more complicated to determine accurately. Hellwarth and Platzman's polaron effective mass m_H is defined by

$$m_H^{-2} = 24 \lim_{\theta \to 0} \kappa\theta \left[\lim_{\lambda \to 0} F(\theta, \lambda)/\lambda^2\right], \quad (55)$$

where κ is Boltzmann's constant. This definition is also an exact prescription since for a weak magnetic field and for low temperature the free energy is determined by the low-lying energy spectrum of a polaron in a magnetic field, and since this spectrum has the same form as the energy spectrum of the motion in a magnetic field of a particle with the self-energy and effective mass of a free polaron. Their result is slightly lower than the Feynman-Schultz result (by at most 1.5%) and may be expressed as

$$m_H^{-2} = (3m_0 - 2m_F)m_0^{-3}, \quad (56)$$

where m_0 and m_F are Feynman's polaron mass as calculated in zeroth and first order in $(S - S')$. For the two-parameter model,

$$m_0 = v^2/w^2. \quad (57)$$

For sake of comparison, the present result may be written in the form

$$\mu'(\alpha)^{-1} = (2m_0 - m_F)\, m_0^{-2}. \quad (58)$$

Hellwarth and Platzman also discuss a generalization of the two-parameter model in which the terms of S' involving the variational parameters C and w are generalized by replacement of C by a variational function $C(w)$ and integration over w. For this generalization, Hellwarth and Platzman point out that Eq. (56) still holds. However, the numerical results would be much more difficult to obtain since the corresponding ground-state energy now has to be minimized with respect to the function $C(w)$. Hellwarth and Platzman further conclude that when the generalized model is fully optimized, $m_0 \to m_F$, which upon use of Eq. (56) yields $m_F = m_0 = m_H$. It

TABLE I. Numerical results for variational parameters, self-energy, and effective mass values.

α	v	w	Polaron self-energy	Polaron effective mass $\mu'(\alpha)$	$m_F(\alpha)$	$\frac{\mu' - m_F}{m_F}$ (%)
1	3.10962	2.87067	−1.01303	1.19594	1.19551	0.04
3	3.42129	2.56030	−3.13333	1.89530	1.88895	0.34
5	4.03434	2.14002	−5.44014	3.91976	3.88562	0.88
7	5.80989	1.60365	−8.11269	14.5298	14.3941	0.94
9	9.85025	1.28230	−11.4858	63.0050	62.7515	0.40
11	15.4132	1.16209	−15.7098	183.433	183.125	0.17
15	30.0822	1.07629	−26.7249	797.845	797.498	0.04

may be added that relation (58) can also be trusted for this generalized model. Therefore, upon full optimization, the generalized model yields $m_F = m_0 = m_H = \mu'$. The method of Feynman's zeroth-order mass calculation can be readily extended to yield

$$m_0 = 1 + 4\int_0^\infty w^{-3} C(w)\, dw\,, \tag{59}$$

for the generalized model.

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APPENDIX

It is to be proved that for a sufficiently weak magnetic field, the inequality (13) holds, where $E_0(\alpha, \lambda)$ and $E_0'(\alpha, \lambda)$ are given by Eqs. (11) and (14).

The path average of any function of path $X[\vec{r}(t)]$, defined by Eq. (17), can be expressed in the alternative form

$$\langle X[\vec{r}(t)]\rangle = \int Xp[\lambda, \vec{r}(t)] D\vec{r}(t)\,, \tag{A1}$$

where

$$p[\lambda, \vec{r}(t)] = e^S \Big/ \int_{\vec{0},0}^{\vec{0},T} e^{S'} D\vec{r}(t)\,. \tag{A2}$$

The main basis of the inequality (13) lies in the fact that for a real variable X, the curve $f_1(X) = e^X$ is always concaved away from the X axis. The tangent to the curve $f_1(X)$ at the point $X = \mathrm{Re}\langle X\rangle$ on the X axis has the equation

$$f_2(X) = e^{\mathrm{Re}\,\langle X\rangle}(X - \mathrm{Re}\,\langle X\rangle + 1)\,, \tag{A3}$$

so that

$$e^X \geq e^{\mathrm{Re}\,\langle X\rangle}(X - \mathrm{Re}\,\langle X\rangle + 1)\,. \tag{A4}$$

Let

$$X = S - S'\,, \tag{A5}$$

(which is a real function of path) and let $I(\lambda)$ be the real function of λ defined by

$$I(\lambda) = \int \{e^X - e^{\mathrm{Re}\,\langle X\rangle}(X - \mathrm{Re}\,\langle X\rangle + 1)\} \times \mathrm{Re}p[\lambda, \vec{r}(t)]\, D\vec{r}(t)\}\,. \tag{A6}$$

Then by multiplying the inequality (A4) by $\mathrm{Re}p \times p[\lambda, \vec{r}(t)]$, path integrating and setting $\lambda = 0$, one obtains

$$I(0) \geq 0 \tag{A7}$$

because $\mathrm{Re}p[0, \vec{r}(t)] = p[0, \vec{r}(t)] \geq 0$. If it is assumed that $I(\lambda)$ is a continuous function of λ at $\lambda = 0$, then $I(\lambda) \geq 0$ for sufficiently small λ. Thus, for a sufficiently weak magnetic field,

$$\int e^X\, \mathrm{Re}p[\lambda, \vec{r}(t)]\, D\vec{r}(t) \geq e^{\mathrm{Re}\langle X\rangle} \int (X - \mathrm{Re}\,\langle X\rangle + 1) \times \mathrm{Re}p[\lambda, \vec{r}(t)]\, D\vec{r}(t)\,. \tag{A8}$$

By use of Eqs. (A1) and (A2), this result can be written in the more compact form

$$\mathrm{Re}\,\langle e^X\rangle \geq e^{\mathrm{Re}\,\langle X\rangle}\,, \tag{A9}$$

which, by use of Eqs. (17) and (A5), becomes

$$\mathrm{Re}\left(\int_{\vec{0},0}^{\vec{0},T} e^S\, D\vec{r}(t) \Big/ \int_{\vec{0},0}^{\vec{0},T} e^{S'} D\vec{r}(t)\right) \geq e^{\mathrm{Re}\,\langle S - S'\rangle}\,. \tag{A10}$$

By taking logarithms of both sides, by using Eqs. (11) and (16) and the fact that both $E_0(\alpha, \lambda)$ and $E_0^{(0)}(\alpha, \lambda)$ are real, and by rearranging terms, one obtains

$$E_0(\alpha, \lambda) \leq E_0^{(0)}(\alpha, \lambda) - \lim_{T\to\infty} T^{-1}\,\mathrm{Re}\,\langle S - S'\rangle\,. \tag{A11}$$

For a sufficiently weak magnetic field, $\langle S - S'\rangle$ was calculated in the text and was found to be real. Hence, by Eq. (14), $E_0(\alpha, \lambda) \leq E_0'(\alpha, \lambda)$ for sufficiently small λ.

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*Based on the Ph.D. thesis of M. S. Chawla, Louisiana State University, Baton Rouge, La., 1971.

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